Appendix to Integrated Distributed Description Logics

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Abstract

This is the proof to the theorem of submitted paper *Integrated Distributed Description Logics*.

Theorem 1 *Reasoning with inference rules of Fig.1 is correct.*

Proof. We prove that for each rule, if an interpretation satisfies the premises, then it satisfies the conclusion. Let $S = \langle \mathbf{O}, \mathbf{A} \rangle$ be a distributed system. Let $\mathcal{I} = \langle \mathbf{I}, \epsilon \rangle$ be a distributed interpretation of S.

1. Let assume that $\mathcal{I} \models_d i : A \sqsubseteq B$

$$\Longrightarrow A^{I_i} \subseteq B^{I_i}$$
$$\Longrightarrow \epsilon_i(A^{I_i}) \subseteq \epsilon_i(B^{I_i})$$
$$\Longrightarrow \mathcal{I} \models_d i : A \xleftrightarrow{\sqsubseteq} i : B$$

2. Let assume that $\mathcal{I} \models_d i : a \sqsubseteq b$

$$\Longrightarrow a^{I_i} = b^{I_i}$$
$$\Longrightarrow \epsilon_i(a^{I_i}) = \epsilon_i(b^{I_i})$$
$$\Longrightarrow \mathcal{I} \models_d i : a \xleftrightarrow{=} i : b$$

3. Let assume that $\mathcal{I} \models_d i : A(a)$

$$\Longrightarrow a^{I_i} \in A^{I_i}$$
$$\Longrightarrow \epsilon_i(a^{I_i}) \in \epsilon_i(A^{I_i})$$
$$\Longrightarrow \mathcal{I} \models_d i : a \xleftrightarrow{\in} i : A$$

4. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\sqsubseteq} j:B \wedge \mathcal{I} \models_d i:A' \xleftrightarrow{\sqsubseteq} j:B'$

$$\Longrightarrow \epsilon_i(A^{I_i}) \subseteq \epsilon_j(B^{I_j}) \wedge \epsilon_i(A'^{I_i}) \subseteq \epsilon_j(B'^{I_j})$$
$$\Longrightarrow \epsilon_i(A^{I_i} \cup A'^{I_i}) \subseteq \epsilon_j(B^{I_j} \cup B'^{I_j})$$
$$\Longrightarrow \mathcal{I} \models_d i:A \sqcup A' \xleftrightarrow{\sqsubseteq} j:B \sqcup B'$$

5. Let assume that $\mathcal{I} \models_d i:a \xleftrightarrow{=} j:b$

$$\Longrightarrow \epsilon_i(a^{I_i}) = \epsilon_j(b^{I_j})$$
$$\Longrightarrow \epsilon_j(b^{I_j}) = \epsilon_i(a^{I_i})$$
$$\Longrightarrow \mathcal{I} \models_d j:b \xleftrightarrow{=} i:a$$

6. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\perp} j:B$

$$\Longrightarrow \epsilon_i(A^{I_i}) \cap \epsilon_j(B^{I_j}) = \emptyset$$
$$\Longrightarrow \epsilon_j(B^{I_j}) \cap \epsilon_i(A^{I_i}) = \emptyset$$
$$\Longrightarrow \mathcal{I} \models_d j:B \xleftrightarrow{\perp} i:A$$

7. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\sqsubseteq} j:B \wedge \mathcal{I} \models_d j:B \xleftrightarrow{\sqsubseteq} k:C$

$$\Longrightarrow \epsilon_i(A^{I_i}) \subseteq \epsilon_j(B^{I_j}) \wedge \epsilon_j(B^{I_j}) \subseteq \epsilon_k(C^{I_k})$$
$$\Longrightarrow \epsilon_i(A^{I_i}) \subseteq \epsilon_k(C^{I_j})$$
$$\Longrightarrow \mathcal{I} \models_d i:A \xleftrightarrow{\sqsubseteq} k:C$$

8. Let assume that $\mathcal{I} \models_d i:a \xleftrightarrow{=} j:b \wedge \mathcal{I} \models_d j:b \xleftrightarrow{\sqsubseteq} k:c$

$$\Longrightarrow \epsilon_i(a^{I_i}) = \epsilon_j(b^{I_j}) \wedge \epsilon_j(b^{I_j}) = \epsilon_k(c^{I_k})$$
$$\Longrightarrow \epsilon_i(a^{I_i}) = \epsilon_k(c^{I_j})$$
$$\Longrightarrow \mathcal{I} \models_d i:a \xleftrightarrow{=} k:c$$

9. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\sqsubseteq} j:B \wedge \mathcal{I} \models_d j:B \xleftrightarrow{\perp} k:C$

$$\Longrightarrow \epsilon_i(A^{I_i}) \subseteq \epsilon_j(B^{I_j}) \wedge \epsilon_j(B^{I_j}) \cap \epsilon_k(C^{I_k}) = \emptyset$$
$$\Longrightarrow \epsilon_i(A^{I_i}) \cap \epsilon_k(C^{I_k}) = \emptyset$$
$$\Longrightarrow \mathcal{I} \models_d i:A \xleftrightarrow{\perp} k:C$$

10. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\perp} j:B \wedge \mathcal{I} \models_d i:A' \xleftrightarrow{\sqsubseteq} j:B$

$$\Longrightarrow \epsilon_i(A^{I_i}) \cap \epsilon_j(B^{I_j}) = \emptyset \wedge \epsilon_i(A'^{I_i}) \subseteq \epsilon_j(B^{I_j})$$
$$\Longrightarrow \epsilon_i(A^{I_i}) \cap \epsilon_i(A'^{I_i}) = \emptyset$$
$$\Longrightarrow A^{I_i} \cap A'^{I_i} = \emptyset$$
$$\Longrightarrow \mathcal{I} \models_d i:A \sqsubseteq \neg A'$$

11. Let assume that $\mathcal{I} \models_d i:A \xleftrightarrow{\perp} j:B \wedge \mathcal{I} \models_d i:a \xleftrightarrow{\in} j:B$

$$\Longrightarrow \epsilon_i(A^{I_i}) \cap \epsilon_j(B^{I_j}) = \emptyset \wedge \epsilon_i(a^{I_i}) \in \epsilon_j(B^{I_j})$$
$$\Longrightarrow \epsilon_i(a^{I_i}) \notin \epsilon_i(A^{I_i})$$
$$\Longrightarrow a^{I_i} \notin A^{I_i}$$
$$\Longrightarrow \mathcal{I} \models_d i:\neg A(a)$$

12. Let assume that $\mathcal{I} \models_d i:a \xleftrightarrow{\in} j:B$

$$\Longrightarrow \epsilon_i(a^{I_i}) \in \epsilon_j(B^{I_j})$$
$$\Longrightarrow \exists \chi \in B^{I_j} \text{ s.t. } \epsilon_j(\chi) = \epsilon_i(a^{I_i})$$

we introduce a new constant x s.t. $x^{I_j} = \chi$.

$$\Longrightarrow \mathcal{I} \models_d j:B(x) \wedge \mathcal{I} \models_d i:a \xleftrightarrow{=} j:x$$

13. Let assume that $\mathcal{I} \models_d i:a \xleftrightarrow{\in} j:B \wedge \mathcal{I} \models_d j:B \xleftrightarrow{\perp} j:B$

$$\Longrightarrow \epsilon_i(a^{I_i}) \in \epsilon_j(B^{I_j}) \wedge \epsilon_j(B^{I_j}) = \emptyset$$
$$\Longrightarrow \Box$$

Theorem 2 *Let $\mathcal{I} = \langle \mathbf{I}, \epsilon \rangle$ be a distributed interpretation of a DS which contains concepts C_i, D_i, roles R_i, S_i, individuals $a_i, b_i, o_1, \ldots, o_n$ in ontology O_i and concept C_j, role R_j, individual a_j in ontology O_j.*

$$I_i \models i:C_i(a_i) \Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow}(a_i^{\rightarrow}) \qquad I_i \models i:C_i \xleftrightarrow{\sqsubseteq} j:C_j \Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq C_j^{\rightarrow}$$
$$I_i \models i:R_i(a_i, b_i) \Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow}(a_i^{\rightarrow}, b_i^{\rightarrow}) \qquad I_i \models i:R_i \xleftrightarrow{\sqsubseteq} j:R_j \Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqsubseteq R_j^{\rightarrow}$$
$$I_i \models i:C_i \sqsubseteq D_i \Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq D_i^{\rightarrow} \qquad I_i \models i:C_i \xleftrightarrow{\perp} j:C_j \Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq \neg(C_j^{\rightarrow})$$
$$I_i \models i:a_i = b_i \Longrightarrow \mathcal{I}^{\rightarrow} \models a_i^{\rightarrow} = b_i^{\rightarrow} \qquad I_i \models i:R_i \xleftrightarrow{\perp} j:R_j \Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqsubseteq \neg(R_j^{\rightarrow})$$
$$I_i \models i:a_i \xleftrightarrow{=} j:a_j \Longrightarrow \mathcal{I}^{\rightarrow} \models a_i^{\rightarrow} = a_j^{\rightarrow} \qquad I_i \models i:a_i \xleftrightarrow{\in} j:C_j \Longrightarrow \mathcal{I}^{\rightarrow} \models C_j^{\rightarrow}(a_i^{\rightarrow})$$

Moreover, the following assertions hold:

$$\mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqcup D_i^{\rightarrow} \sqsubseteq (C_i \sqcup D_i)^{\rightarrow} \qquad \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqcup S_i^{\rightarrow} \sqsubseteq (R_i \sqcup S_i)^{\rightarrow}$$
$$\mathcal{I}^{\rightarrow} \models (C_i \sqcup D_i)^{\rightarrow} \sqsubseteq C_i^{\rightarrow} \sqcup D_i^{\rightarrow} \qquad \mathcal{I}^{\rightarrow} \models (R_i \sqcup S_i)^{\rightarrow} \sqsubseteq R_i^{\rightarrow} \sqcup S_i^{\rightarrow}$$
$$\mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqcap D_i^{\rightarrow} \sqsubseteq (C_i \sqcap D_i)^{\rightarrow} \qquad \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqcap S_i^{\rightarrow} \sqsubseteq (R_i \sqcap S_i)^{\rightarrow}$$
$$\mathcal{I}^{\rightarrow} \models (\exists R_i.C_i)^{\rightarrow} \sqsubseteq \exists (R_i^{\rightarrow}).(C_i^{\rightarrow}) \qquad \mathcal{I}^{\rightarrow} \models (R_i^{-})^{\rightarrow} \sqsubseteq (R_i^{\rightarrow})^{-}$$
$$\mathcal{I}^{\rightarrow} \models \exists R_i^{\rightarrow}.\top \sqsubseteq (\exists R_i.\top)^{\rightarrow} \qquad \mathcal{I}^{\rightarrow} \models (R_i^{\rightarrow})^{-} \sqsubseteq (R_i^{-})^{\rightarrow}$$
$$\mathcal{I}^{\rightarrow} \models (\{o_1, \ldots, o_n\})^{\rightarrow} \sqsubseteq \{o_1^{\rightarrow}, \ldots, o_n^{\rightarrow}\} \qquad \mathcal{I}^{\rightarrow} \models (R_i^{+})^{\rightarrow} \sqsubseteq (R_i^{\rightarrow})^{+}$$
$$\mathcal{I}^{\rightarrow} \models \{o_1^{\rightarrow}, \ldots, o_n^{\rightarrow}\} \sqsubseteq \{(o_1, \ldots, o_n\})^{\rightarrow} \qquad \mathcal{I}^{\rightarrow} \models (R_i \circ S_i)^{\rightarrow} \sqsubseteq R_i^{\rightarrow} \circ S_i^{\rightarrow}$$

Proof.

- Let us assume that $I_i \models i:C_i(a_i)$

$$\Longrightarrow a_i^{I_i} \in C_i^{I_i}$$
$$\Longrightarrow \epsilon_i(a_i^{I_i}) \in \epsilon_i(C_i^{I_i})$$
$$\Longrightarrow (a_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \in (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$$
$$\Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow}(a_i^{\rightarrow})$$

- Let us assume that $I_i \models i\!:\!R_i(a_i, b_i)$

$$\begin{aligned}
&\Longrightarrow \langle a_i^{I_i}, b_i^{I_i} \rangle \in R_i^{I_i} \\
&\Longrightarrow \langle \epsilon_i(a_i^{I_i}), \epsilon_i(b_i^{I_i}) \rangle \in \epsilon_i(R_i^{I_i}) \\
&\Longrightarrow \langle (a_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}, (b_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \rangle \in (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow}(a_i^{\rightarrow}, b_i^{\rightarrow})
\end{aligned}$$

- Let us assume that $I_i \models i\!:\!C_i \sqsubseteq D_i$

$$\begin{aligned}
&\Longrightarrow C_i^{I_i} \subseteq D_i^{I_i} \\
&\Longrightarrow \epsilon_i(C_i^{I_i}) \subseteq \epsilon_i(D_i^{I_i}) \\
&\Longrightarrow (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \subseteq (D_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq D_i^{\rightarrow}
\end{aligned}$$

- Let us assume that $I_i \models i\!:\!a_i = b_i$

$$\begin{aligned}
&\Longrightarrow a_i^{I_i} = b_i^{I_i} \\
&\Longrightarrow \epsilon_i(a_i^{I_i}) = \epsilon_i(b_i^{I_i}) \\
&\Longrightarrow (a_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = (b_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models a_i^{\rightarrow} = b_i^{\rightarrow}
\end{aligned}$$

- Let us assume that $I_i \models i\!:\!a_i \xleftrightarrow{=} j\!:\!a_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(a_i^{I_i}) = \epsilon_j(a_j^{I_j}) \\
&\Longrightarrow (a_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = (a_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models a_i^{\rightarrow} = a_j^{\rightarrow}
\end{aligned}$$

- Let us assume that $I_i \models i\!:\!C_i \xleftrightarrow{\sqsubseteq} j\!:\!C_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(C_i^{I_i}) \subseteq \epsilon_j(C_j^{I_j}) \\
&\Longrightarrow (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = (C_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq C_j^{\rightarrow}
\end{aligned}$$

- Let us assume that $I_i \models i\!:\!R_i \xleftrightarrow{\sqsubseteq} j\!:\!R_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(R_i^{I_i}) \subseteq \epsilon_j(R_j^{I_j}) \\
&\Longrightarrow (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \subseteq (R_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqsubseteq R_j^{\rightarrow}
\end{aligned}$$

- Let us assume that $I_i \models i : C_i \xleftrightarrow{\perp} j : C_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(C_i^{I_i}) \cap \epsilon_j(C_j^{I_j}) = \emptyset \\
&\Longrightarrow (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \cap (C_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqsubseteq \neg(C_j^{\rightarrow})
\end{aligned}$$

- Let us assume that $I_i \models i : R_i \xleftrightarrow{\perp} j : R_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(R_i^{I_i}) \cap \epsilon_j(R_j^{I_j}) = \emptyset \\
&\Longrightarrow (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \cap (R_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqsubseteq \neg(R_j^{\rightarrow})
\end{aligned}$$

- Let us assume that $I_i \models i : a_i \xleftrightarrow{\in} j : C_j$

$$\begin{aligned}
&\Longrightarrow \epsilon_i(a_i^{I_i}) \in \epsilon_j(C_j^{I_j}) \\
&\Longrightarrow (a_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \in (C_j^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&\Longrightarrow \mathcal{I}^{\rightarrow} \models C_j^{\rightarrow}(a_i^{\rightarrow})
\end{aligned}$$

Other proofs:

- In the following proof, C_i and D_i can be replaced by R_i and S_i respectively to obtain the proof for roles.

$$\begin{aligned}
(C_i^{\rightarrow} \sqcup D_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} &= (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \cup (D_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&= \epsilon_i(C_i^{I_i}) \cup \epsilon_i(D_i^{I_i}) \\
&= \epsilon_i(C_i^{I_i} \cup D_i^{I_i}) \\
&= \epsilon_i((C_i \sqcup D_i)^{I_i}) \\
&= ((C_i \sqcup D_i)^{\rightarrow})^{\mathcal{I}^{\rightarrow}}
\end{aligned}$$

  Therefore $\mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqcup D_i^{\rightarrow} \sqsubseteq (C_i \sqcup D_i)^{\rightarrow}$, $\mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqcup S_i^{\rightarrow} \sqsubseteq (R_i \sqcup S_i)^{\rightarrow}$, $\mathcal{I}^{\rightarrow} \models (C_i \sqcup D_i)^{\rightarrow} \sqsubseteq C_i^{\rightarrow} \sqcup D_i^{\rightarrow}$ and $\mathcal{I}^{\rightarrow} \models (R_i \sqcup S_i)^{\rightarrow} \sqsubseteq R_i^{\rightarrow} \sqcup S_i^{\rightarrow}$.

- Again, C_i and D_i can be replaced by R_i and S_i.

$$\begin{aligned}
(C_i^{\rightarrow} \sqcap D_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} &= (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \cap (D_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} \\
&= \epsilon_i(C_i^{I_i}) \cap \epsilon_i(D_i^{I_i}) \\
&\subseteq \epsilon_i(C_i^{I_i} \cap D_i^{I_i}) \\
&= \epsilon_i((C_i \sqcap D_i)^{I_i}) \\
&= ((C_i \sqcap D_i)^{\rightarrow})^{\mathcal{I}^{\rightarrow}}
\end{aligned}$$

  Therefore $\mathcal{I}^{\rightarrow} \models C_i^{\rightarrow} \sqcap D_i^{\rightarrow} \sqsubseteq (C_i \sqcap D_i)^{\rightarrow}$ and $\mathcal{I}^{\rightarrow} \models R_i^{\rightarrow} \sqcap S_i^{\rightarrow} \sqsubseteq (R_i \sqcap S_i)^{\rightarrow}$.

- Let $x \in ((\exists R_i C_i)^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. There exists $x' \in (\exists R_i C_i)^{I_i}$ s.t. $\epsilon_i(x') = x$. There exists $y' \in C_i^{I_i}$ s.t. $\langle x', y' \rangle \in R_i^{I_i}$. $\epsilon_i(y') \in \epsilon_i(C_i^{I_i}) = (C_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$ and $\langle \epsilon_i(x'), \epsilon_i(y') \rangle \in \epsilon(R_i^{I_i}) = (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. So $\epsilon_i(x') \in (\exists (R_i^{\rightarrow})(C_i^{\rightarrow}))^{\mathcal{I}^{\rightarrow}}$. Hence $\mathcal{I}^{\rightarrow} \models (\exists R_i.C_i)^{\rightarrow} \sqsubseteq \exists (R_i^{\rightarrow}).(C_i^{\rightarrow})$.

- Here, R_i is a role:

$$
\begin{aligned}
((R_i^{-})^{\rightarrow})^{\mathcal{I}^{\rightarrow}} &= \epsilon_i((R_i^{-})^{I_i}) \\
&= \epsilon_i(\{\langle y, x \rangle ; \langle x, y \rangle \in R_i^{I_i}\}) \\
&= \{\langle v, u \rangle ; \langle u, v \rangle \in \epsilon_i(R_i^{I_i})\} \\
&= ((R_i^{\rightarrow})^{-})^{\mathcal{I}^{\rightarrow}}
\end{aligned}
$$

  Therefore $\mathcal{I}^{\rightarrow} \models (R_i^{-})^{\rightarrow} \sqsubseteq (R_i^{\rightarrow})^{-}$ and $\mathcal{I}^{\rightarrow} \models (R_i^{\rightarrow})^{-} \sqsubseteq (R_i^{-})^{\rightarrow}$.

- Let $x \in (\exists R_i^{\rightarrow}.\top)^{\mathcal{I}^{\rightarrow}}$. There exists y s.t. $\langle x, y \rangle \in (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \epsilon_i(R_i^{I_i})$. So there exists $\langle x', y' \rangle \in R_i^{I_i}$ s.t. $\epsilon_i(x') = x$ and $\epsilon_i(y') = y$. So $x' \in (\exists R_i.\top)^{I_i}$. Therefore, $x \in ((\exists.\top)^{I_i})^{\mathcal{I}^{\rightarrow}}$. Hence, $\mathcal{I}^{\rightarrow} \models \exists R_i^{\rightarrow}.\top \sqsubseteq (\exists R_i.\top)^{\rightarrow}$.

- And here is the proof for the nominals:

$$
\begin{aligned}
(\{o_1, \ldots, o_n\}^{\rightarrow})^{\mathcal{I}^{\rightarrow}} &= \epsilon_i(\{o_1, \ldots, o_n\}^{I_i}) \\
&= \epsilon_i(\{o_1^{I_i}, \ldots, o_n^{I_i}\}) \\
&= \{\epsilon_i(o_1^{I_i}), \ldots, \epsilon_i(o_n^{I_i})\} \\
&= \{(o_1^{\rightarrow})^{\mathcal{I}^{\rightarrow}}, \ldots, (o_n^{\rightarrow})^{\mathcal{I}^{\rightarrow}}\}
\end{aligned}
$$

  Therefore $\mathcal{I}^{\rightarrow} \models (\{o_1, \ldots, o_n\})^{\rightarrow} \sqsubseteq \{o_1^{\rightarrow}, \ldots, o_n^{\rightarrow}\}$ and $\mathcal{I}^{\rightarrow} \models \{o_1^{\rightarrow}, \ldots, o_n^{\rightarrow}\} \sqsubseteq \{(o_1, \ldots, o_n\})^{\rightarrow}$.

- Let $\langle x, y \rangle \in ((R_i^{+})^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. There exist $\langle x', y' \rangle \in (R_i^{+})^{I_i}$ s.t. $\epsilon_i(x') = x$ and $\epsilon_i(y') = y$. So there exists $x_1, \ldots, x_k \in \Delta^{I_i}$ s.t. $\langle x', x_1 \rangle \in R_i^{I_i}, \langle x_1, x_2 \rangle \in R_i^{I_i}, \ldots, \langle x_k, y' \rangle \in R_i^{I_i}$. So $\langle \epsilon_i(x'), \epsilon_i(x_1) \rangle \in \epsilon_i(R_i^{I_i}), \langle \epsilon_i(x_1), \epsilon_i(x_2) \rangle \in \epsilon_i(R_i^{I_i}), \ldots, \langle \epsilon_i(x_k), \epsilon_i(y') \rangle \in \epsilon_i(R_i^{I_i})$. So $\langle \epsilon_i(x'), \epsilon_i(y') \rangle \in (R_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. Therefore $\mathcal{I}^{\rightarrow} \models (R_i^{+})^{\rightarrow} \sqsubseteq (R_i^{\rightarrow})^{+}$.

- Let $\langle x, y \rangle \in ((R_i \circ S_i)^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. There exist $\langle x', y' \rangle \in (R_i \circ S_i)^{I_i}$ s.t. $\epsilon_i(x') = x$ and $\epsilon_i(y') = y$. So there exists $z' \in \Delta^{I_i}$ s.t. $\langle x', z' \rangle \in R_i^{I_i}$ and $\langle z', y' \rangle \in S_i^{I_i}$. So $\langle \epsilon_i(x'), \epsilon_i(z') \rangle \in \epsilon_i(R_i^{I_i})$ and $\langle \epsilon_i(x'), \epsilon_i(z') \rangle \in \epsilon_i(S_i^{I_i})$. Therefore, $\langle \epsilon_i(x'), \epsilon_i(y') \rangle \in (R_i^{\rightarrow} \circ S_i^{\rightarrow})^{\mathcal{I}^{\rightarrow}}$. Hence, $\mathcal{I}^{\rightarrow} \models (R_i \circ S_i)^{\rightarrow} \sqsubseteq R_i^{\rightarrow} \circ S_i^{\rightarrow}$.

Counter examples for other constructors:

Theorem 3 *Let O_i be an ontology in a distributed system S. Let C and D be two concepts of O_i, R and S be two roles of O_i, n a nonzero natural number. For each of the following formulas, there is a distributed interpretation $\mathcal{I}$ such* *that $\mathcal{I}^{\rightarrow}$ does not satisfy the formula.*

$$(\forall R.C)^{\rightarrow} \sqsubseteq \forall R^{\rightarrow}.C^{\rightarrow} \quad (1)$$
$$\forall R^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\forall R.C)^{\rightarrow} \quad (2)$$
$$\exists R^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\exists R.C)^{\rightarrow} \quad (3)$$
$$C^{\rightarrow} \sqcap D^{\rightarrow} \sqsubseteq (C \sqcap D)^{\rightarrow} \quad (4)$$
$$R^{\rightarrow} \sqcap S^{\rightarrow} \sqsubseteq (R \sqcap S)^{\rightarrow} \quad (5)$$
$$(\leq nR.C)^{\rightarrow} \sqsubseteq \leq nR^{\rightarrow}.C^{\rightarrow} \quad (6)$$
$$\leq nR^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\leq nR.C)^{\rightarrow} \quad (7)$$
$$(\geq nR.C)^{\rightarrow} \sqsubseteq \geq nR^{\rightarrow}.C^{\rightarrow} \quad (8)$$
$$\geq nR^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\geq nR.C)^{\rightarrow} \quad (9)$$
$$(R^{\rightarrow})^{+} \sqsubseteq (R^{+})^{\rightarrow} \quad (10)$$
$$R^{\rightarrow} \circ S^{\rightarrow} \sqsubseteq (R \circ S)^{\rightarrow} \quad (11)$$

For formula (8) and (9) n must be greater or equal to 2.

Proof. We give counter example for each of the previous formulas:

- (1) and (2): Consider the local interpretation I with domain $\Delta^I = \{u, v, w, x, y, z\}$, and interpretation function such that $C^I = \{v\}$, $R^I = \{\langle u, v\rangle, \langle w, x\rangle, \langle y, z\rangle\}$. Consider also the equalizing function $\epsilon : \Delta^I \to \Delta$ with $\Delta = \{a, b, c, d\}$ and s.t. $\epsilon(u) = \epsilon(w) = a$, $\epsilon(v) = \epsilon(z) = b$, $\epsilon(x) = c$, and $\epsilon(y) = d$. Then, $(\forall R.C)^I = \{u\}$, and $((\forall R.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$, while $((\forall R^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{d\}$. So $\mathcal{I}^{\rightarrow} \not\models_d (\forall R.C)^{\rightarrow} \sqsubseteq \forall R^{\rightarrow}.C^{\rightarrow}$ and $\mathcal{I}^{\rightarrow} \not\models_d \forall R^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\forall R.C)^{\rightarrow}$.

- (3): With the same interpretation as in previous item, $(\exists R.C)^I = \{u\}$, and $((\exists R.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$. But $(\exists R^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a, d\}$. So $\mathcal{I}^{\rightarrow} \not\models_d \exists R^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\exists R.C)^{\rightarrow}$.

- (4): Consider the local interpretation I with domain $\Delta^I = \{y, z\}$, and interpretation function such that $C^I = \{y\}$, $D^I = \{z\}$. Consider also the equalizing function $\epsilon : \Delta^I \to \Delta$ with $\Delta = \{a\}$ and s.t. $\epsilon(y) = \epsilon(z) = a$. Then, $(C \sqcap D)^I = \emptyset$, and $((C \sqcap D)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$, while $(C^{\rightarrow} \sqcap D^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$. So $\mathcal{I}^{\rightarrow} \not\models_d C^{\rightarrow} \sqcap D^{\rightarrow} \sqsubseteq (C \sqcap D)^{\rightarrow}$.

- (5): Consider the local interpretation I with domain $\Delta^I = \{w, x, y, z\}$, and interpretation function such that $R^I = \{\langle w, x\rangle\}$, $S^I = \{\langle y, z\rangle\}$. Consider also the equalizing function $\epsilon : \Delta^I \to \Delta$ with $\Delta = \{a, b\}$ and s.t. $\epsilon(w) = \epsilon(y) = a$ and $\epsilon(w) = \epsilon(z) = b$. Then, $(R \sqcap S)^I = \emptyset$, and $((R \sqcap S)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$, while $(R^{\rightarrow} \sqcap S^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{\langle a, b\rangle\}$. So $\mathcal{I}^{\rightarrow} \not\models_d R^{\rightarrow} \sqcap S^{\rightarrow} \sqsubseteq (R \sqcap S)^{\rightarrow}$.

- (6): Consider the local interpretation I with domain $\Delta^I = \{x, y, \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n\}$, and interpretation function such that $R^I = \{\langle x, \lambda_1\rangle, \dots, \langle x, \lambda_n\rangle, \langle y, \mu_1\rangle, \dots, \langle y, \mu_n\rangle\}$ and $C^I = \{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n\}$. Consider also the equalizing function $\epsilon : \Delta^I \to \Delta$ with $\Delta = \{a, b_1, \dots, b_n, c_1, \dots, c_n\}$ and s.t. $\epsilon(x) = \epsilon(y) = a$, $\epsilon(\lambda_i) = b_i$ and $\epsilon(\mu_i) = c_i$. Then, $(\leq nR.C)^I = \{x, y\}$, and

$((\leq nR.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$, while $(\leq nR^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$. So $\mathcal{I}^{\rightarrow} \not\models_d (\leq nR.C)^{\rightarrow} \sqsubseteq \leq nR^{\rightarrow}.C^{\rightarrow}$.

- (7): Consider the local interpretation I with domain $\Delta^I = \{x, y, z, \lambda_1, \ldots, \lambda_{n+1}\}$, and interpretation function such that $R^I = \{\langle x, y\rangle, \langle z, \lambda_1\rangle, \ldots, \langle z, \lambda_{n+1}\rangle\}$ and $C^I = \{y, \lambda_1, \ldots, \lambda_{n+1}\}$. Consider also the equalizing function $\epsilon : \Delta^I \rightarrow \Delta$ with $\Delta = \{a, b, c\}$ and s.t. $\epsilon(x) = a$, $\epsilon(y) = \epsilon(\lambda_i) = b$ and $\epsilon(z) = c$. Then, $(\leq nR.C)^I = \{x\}$, and $((\leq nR.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$, while $(\leq nR^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a, c\}$. So $\mathcal{I}^{\rightarrow} \not\models_d \leq nR^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\leq nR.C)^{\rightarrow}$.

- (8): Let us assume that $n \geq q2$. Consider the local interpretation I with domain $\Delta^I = \{x, \lambda_1, \ldots, \lambda_{n+1}\}$, and interpretation function such that $R^I = \{\langle x, \lambda_1\rangle, \ldots, \langle x, \lambda_{n+1}\rangle\}$ and $C^I = \{\lambda_1, \ldots, \lambda_{n+1}\}$. Consider also the equalizing function $\epsilon : \Delta^I \rightarrow \Delta$ with $\Delta = \{a, b\}$ and s.t. $\epsilon(x) = a$ and $\epsilon(\lambda_i) = b$. Then, $(\geq nR.C)^I = \{x\}$, and $((\geq nR.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$, while $(\geq nR^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$ because $n \geq 2$. So $\mathcal{I}^{\rightarrow} \not\models_d (\geq nR.C)^{\rightarrow} \sqsubseteq \geq nR^{\rightarrow}.C^{\rightarrow}$.

- (9): Let us assume that $n \geq q2$. Consider the local interpretation I with domain $\Delta^I = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and interpretation function such that $R^I = \{\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle\}$ and $C^I = \{y_1, \ldots, y_n\}$. Consider also the equalizing function $\epsilon : \Delta^I \rightarrow \Delta$ with $\Delta = \{a, b_1, \ldots, b_n\}$ and s.t. $\epsilon(x_i) = a$ and $\epsilon(y_i) = b_i$. Then, $(\geq nR.C)^I = \emptyset$ because $n \geq 2$, and $((\geq nR.C)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$, while $(\geq nR^{\rightarrow}.C^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{a\}$. So $\mathcal{I}^{\rightarrow} \not\models_d \geq nR^{\rightarrow}.C^{\rightarrow} \sqsubseteq (\geq nR.C)^{\rightarrow}$.

- (10): Consider the local interpretation I with domain $\Delta^I = \{w, x, y, z\}$, and interpretation function such that $R^I = \{\langle w, x\rangle, \langle y, z\rangle\}$. Consider also the equalizing function $\epsilon : \Delta^I \rightarrow \Delta$ with $\Delta = \{a, b, c\}$ and s.t. $\epsilon(w) = a$, $\epsilon(x) = \epsilon(y) = b$ and $\epsilon(z) = c$. Then, $(R^+)^I = \{\langle w, x\rangle, \langle y, z\rangle\}$, and $((R^+)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{\langle a, b\rangle, \langle b, c\rangle\}$, while $((R^{\rightarrow})^+)^{\mathcal{I}^{\rightarrow}} = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$. So $\mathcal{I}^{\rightarrow} \not\models_d (R^{\rightarrow})^+ \sqsubseteq (R^+)^{\rightarrow}$.

- (11): Consider the local interpretation I with domain $\Delta^I = \{w, x, y, z\}$, and interpretation function such that $R^I = \{\langle w, x\rangle\}$ and $S^I = \{\langle y, z\rangle\}$. Consider also the equalizing function $\epsilon : \Delta^I \rightarrow \Delta$ with $\Delta = \{a, b, c\}$ and s.t. $\epsilon(w) = a$, $\epsilon(x) = \epsilon(y) = b$ and $\epsilon(z) = c$. Then, $(R \circ S)^I = \emptyset$, and $((R \circ S)^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \emptyset$, while $(R^{\rightarrow} \circ S^{\rightarrow})^{\mathcal{I}^{\rightarrow}} = \{\langle a, b\rangle, \langle b, c\rangle\}$. So $\mathcal{I}^{\rightarrow} \not\models_d R^{\rightarrow} \circ S^{\rightarrow} \sqsubseteq (R \circ S)^{\rightarrow}$.